

Fock-Schwinger proper time formalism for p -branes¹

Matej Pavšič

Jožef Stefan Institute, University of Ljubljana, Jamova 39, 1000 Ljubljana, Slovenia²

ABSTRACT

The theory of the usual, constrained p -branes is embedded into a larger theory in which there is no constraints. In the latter theory the Fock-Schwinger proper time formalism is extended from point-particles to p -branes which can be considered as points in an infinite dimensional space \mathcal{M} . The quantization appears to be straightforward and elegant. The conventional p -brane states are particular stationary solutions to the functional Schrödinger equation which describes the evolution of a membrane's state with respect to the invariant evolution parameter τ . It is also shown that states of a lower dimensional p -brane can be considered as particular states of a higher dimensional p -brane.

1 Introduction

Relativistic p -branes are being intensively studied nowadays [1]. A very interesting mathematical structure is being revealed and many are convinced that, in one way or another, it will find its place in physics. A consistent quantum theory of a p -brane can be formulated in the embedding space of a certain characteristic dimension (e.g. 26 for bosonic strings). The problem then occurs how to compactify all those extra dimensions to the observed 4 dimensions. We prefer to adopt a different view and assume that a 3-brane sweeping a 4-dimensional worldsheet V_4 in an embedding space V_N (where N is determined by consistency conditions) already represents a spacetime [2, 3]. There is no need to compactify the dimensions of V_N . But, unfortunately, quantization of such a higher dimensional extended object is extremely involved because of the presence of the constraints due to the reparametrization invariance. As a way to avoid such difficulties I proposed to consider the so called unconstrained p -branes [4]. The latter objects I shall often call simply "membranes" (their dimensionality not being restricted to 2). Arbitrary deformations of a membrane are allowed and there is no constraints. Quantization of such a system is straightforward, and the conventional p -brane states occur as stationary solutions to the Schrödinger equation.

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² E-mail: MATEJ.PAVSIC@IJS.SI

2 The classical dynamics of unconstrained membranes

We shall first consider the classical dynamics of an n -dimensional membrane \mathcal{V}_n described by the variables $X^\mu(\tau, \xi^a)$, $\mu = 1, 2, \dots, N$; $a = 1, 2, \dots, n$ which denote position of \mathcal{V}_n in V_N . We assume the following action which is an extension of the Stueckelberg action from point particles to membranes:

$$I = \frac{\kappa}{2} \int d\tau d^n\xi \sqrt{|f|} \left[\frac{1}{\Lambda} g_{\mu\nu} (\dot{X}^\mu + \Lambda^a \partial_a X^\mu) (\dot{X}^\nu + \Lambda^b \partial_b X^\nu) + \Lambda \right] \quad (1)$$

where $f \equiv \det f_{ab}$ is the determinant of the induced metric $f_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$ on \mathcal{V}_n , κ a constant and $g_{\mu\nu}$ the metric of V_N . Variation of the action in our approach is to be performed solely with respect to X^μ , while Λ and Λ^a are assumed to be fixed background fields on \mathcal{V}_n . In the conventional approach Λ , Λ^a are Lagrange multipliers leading to the p -brane constraints and the action (1) is equivalent to the Dirac-Nambu-Goto action; fixing of Λ , Λ^a then fixes a gauge. On the contrary, in our approach Λ , Λ^a are not Lagrange multipliers at all, they are fixed from the very beginning: a physical consequence is that a membrane is arbitrarily deformable, and its tension is not necessarily a constant, but depends on a solution of the equations of motion [3, 4].

The quantities $X^\mu(\tau, \xi)$ are independent dynamical variables, and there is no constraints. Those n components among $X^\mu(\tau, \xi)$, roughly speaking, which are redundant in the conventional approach and must be determined by choice of a gauge, are not redundant at all in our approach: they are necessary to describe deformations of membrane [3, 4].

The equation of motion derived from (1) gives after contraction with $\partial^c X^\mu$ and rearrangement of the terms:

$$\frac{d}{d\tau} (p_\mu \partial_c X^\mu) + \partial_a (\Lambda^a p_\mu \partial_c X^\mu) = \frac{1}{2} \partial_c \Lambda \frac{\sqrt{|f|}}{\kappa} \left(\frac{p^2}{|f|} - \kappa^2 \right) \quad (2)$$

where $p_\mu = (\kappa \sqrt{|f|}/\Lambda) (\dot{X}_\mu + \Lambda^a \partial_a X_\mu)$ is the canonical momentum. Eq.(2) admits a solution satisfying

$$p_\mu \partial_c X^\mu = 0 \quad (3)$$

$$p^2 - |f| \kappa^2 = 0 \quad (4)$$

which are just the p -brane constraints. Conventional p -branes, satisfying the minimal surface equation are among possible solutions to our dynamical system. When $\Lambda^a = 0$, Eq.(3) says that $\dot{X}^\mu \partial_a X_\mu = 0$ which means that the velocity \dot{X}^μ is perpendicular to the membrane. But in general, $p_\mu \partial_c X^\mu \neq 0$, and the velocity has non zero tangent component to the membrane, so that different parts of the membrane move relative to each other. Such a membrane is then a wiggly membrane [5].

Different functions $X^\mu(\xi)$ and $X'^\mu(\xi)$ which happen to describe the same surface V_n we can interpret

(i) *passively* (as representing the same membrane \mathcal{V}_n in different parametrizations of ξ^a),

(ii) *actively* (as representing two physically distinct, deformed, membranes \mathcal{V}_n and \mathcal{V}'_n , respectively).

In order to better understand this, just imagine a rubber sheet spanning a surface V_2 . It may happen that two physically different configurations $\mathcal{V}_2, \mathcal{V}'_2$ of the deformed sheet span the same surface V_2 .

3 The membrane space

It is convenient to introduce the concept of *membrane space* \mathcal{M} the points of which are unconstrained membranes \mathcal{V}_n , parametrized by coordinates $X^\mu(\xi) \equiv X^{\mu(\xi)}$. The distance is defined by

$$d\ell^2 = \rho_{\mu(\xi)\nu(\zeta)} dX^{\mu(\xi)} dX^{\nu(\zeta)} = dX^{\mu(\xi)} dX_{\mu(\xi)} \quad (5)$$

where the metric is

$$\rho_{\mu(\xi)\nu(\zeta)} = \frac{\kappa\sqrt{|f|}}{\Lambda} g_{\mu\nu} \delta(\xi - \zeta) \quad (6)$$

In Eq.(5) we adopt the convention of summation over repeated indices, such as μ, ν and integration over the repeated continuous indices, such as ξ, ξ' .

The tensor calculus in \mathcal{M} [3] is a straightforward generalization of the tensor calculus in a finite dimensional space. From the metric (6) and its inverse

$$\rho^{\mu(\xi)\nu(\xi')} = \frac{\Lambda}{\kappa\sqrt{|f|}} g^{\mu\nu} \delta(\xi - \xi') \quad (7)$$

we can construct the affinity $\Gamma_{\alpha(\xi')\beta(\xi'')}^{\mu(\xi)}$ and define the covariant (functional) derivative $D/DX^{\mu(\xi)} \equiv D_{\mu(\xi)}$ in \mathcal{M} . When acting on a scalar functional $A[X(\xi)]$ the covariant derivative coincides with the partial functional derivative $\delta/\delta X^\mu(\xi) \equiv \partial/\partial X^{\mu(\xi)} \equiv \partial_{\mu(\xi)}$. But in general, a geometric object in \mathcal{M} is a tensor of arbitrary rank, $A^{\mu(\xi)}_{\nu(\xi')\dots}$ which is a function of $X^{\mu(\xi)}$ (i.e., a functional of $X^\mu(\xi)$).

The invariant measure in \mathcal{M} (i.e., the volume element) is

$$\mathcal{D}X = \left(\text{Det}\rho_{\mu(\xi)\nu(\xi')}\right)^{1/2} \prod_{\mu,\xi} dX^{\mu(\xi)} = \prod_{\mu,\xi} \left(\frac{\kappa\sqrt{|f|}}{\Lambda}\right)^{1/2} dX^{\mu(\xi)} \quad (8)$$

The action (1) can be written in the compact notation of \mathcal{M} space as

$$I = \frac{1}{2} \int d\tau \left[\frac{1}{\Lambda} \rho_{\mu(\xi)\nu(\xi')} (\dot{X}^{\mu(\xi)} + \Lambda^a \partial_a X^{\mu(\xi)}) (\dot{X}^{\nu(\xi')} + \Lambda^b \partial_b X^{\nu(\xi')}) + K \right] \quad (9)$$

where $K \equiv \int d^n \xi \sqrt{|f|} \Lambda \kappa$. When $\Lambda^a = 0$ Eq.(9) has the same form as the well known Stueckelberg point-particle action [6]. However, the space in which the dynamics takes place is now the infinite dimensional membrane space \mathcal{M} , and the evolution parameter τ is essentially the proper time in \mathcal{M} . The spacetime, which is the arena for the dynamics of a Stueckelberg point particle, is just a subspace of \mathcal{M} . The action (9) is a generalization of the Stueckelberg action and it describes membranes of arbitrary dimensions n , including point-particles, when $n = 0$.

The *Hamiltonian* belonging to (1) is

$$H = \int d^n \xi \left[\sqrt{|f|} \frac{\Lambda}{2\kappa} \left(\frac{p^\mu p_\mu}{|f|} - \kappa^2 \right) - \Lambda^a \partial_a X^\mu p_\mu \right] \quad (10)$$

and can be written compactly as

$$H = \frac{1}{2} (p^{\mu(\xi)} p_{\mu(\xi)} - K) - \Lambda^a \partial_a X^{\mu(\xi)} p_{\mu(\xi)} \quad (11)$$

where and $p_{\mu(\xi)} \equiv p_\mu(\xi)$. The canonical and the Hamilton-Jacobi theory for unconstrained membranes can be straightforwardly developed [8].

4 The quantum theory of unconstrained membranes

Quantization of the theory is performed by considering $X^\mu(\xi)$ and $p_\mu(\xi)$ as operators satisfying the commutation relations

$$[X^\mu(\xi), p_\nu(\xi')] = i \delta^\mu_\nu \delta(\xi - \xi') \quad (12)$$

$$[X^\mu(\xi), X^\nu(\xi')] = 0, \quad [p_\mu(\xi), p_\nu(\xi')] = 0 \quad (13)$$

In the coordinate representation $X^\mu(\xi)$ are diagonal. The momentum operator is $p_\mu(\xi) = -i\delta/\delta X^\mu(\xi)$, when acting on a scalar functional; otherwise it is given by the covariant derivative $p_\mu(\xi) \equiv p_{\mu(\xi)} = -i\mathcal{D}_{\mu(\xi)}$. A state is represented by a τ -dependent wave functional $\psi[\tau, X^\mu(\xi)]$ satisfying the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial \tau} = H\psi \quad (14)$$

where H is given by (11) with $p_{\mu(\xi)} = -i\mathcal{D}_{\mu(\xi)}$. Eq.(14) is a generalization of the well known point-particle relativistic wave equation with proper time [6].

The wave functional ψ is normalized according to [6]

$$\int \mathcal{D}X \psi^* \psi = 1 \quad (15)$$

where $\mathcal{D}X$ is given in (8). Eq.(15) is a straightforward extension of the corresponding relation $\int d^4x \psi^* \psi = 1$ for the unconstrained point particle in Minkowski spacetime [6, 7]. It is important to stress that, since (15) is satisfied at any τ , the evolution operator U which brings $\psi(\tau) \rightarrow \psi(\tau') = U\psi(\tau)$ is *unitary*, and no negative norm states occur in such a theory.

Stationary solutions to the Schrödinger equation (14) are given by

$$\psi[\tau, X^\mu(\xi)] = e^{-iE\tau} \phi[X^\mu(\xi)] \quad (16)$$

where E is a constant of motion, and satisfy

$$\left(-\frac{1}{2} \mathcal{D}^{\mu(\xi)} \mathcal{D}_{\mu(\xi)} + i \Lambda^a \partial_a X^{\mu(\xi)} \mathcal{D}_{\mu(\xi)} - \frac{1}{2} K \right) \phi = E \phi \quad (17)$$

For a real ϕ Eq.(17) splits into

$$\left(-\frac{1}{2}D^{\mu(\xi)}D_{\mu(\xi)} - \frac{1}{2}K - E\right)\phi = 0 \quad (18)$$

$$\Lambda^a \partial_a X^{\mu(\xi)} D_{\mu(\xi)} \phi = 0 \quad (19)$$

The latter equations are satisfied by the well known p -brane constraints acting on a state ϕ . We see that our Schrödinger equation (14) contains the conventional p -brane states as particular solutions.

Wave packets of membrane's states can also be studied with our approach, at least in a simple case when the centre of the wave packet has linear dependence on τ [3]. This corresponds to the case of a null string [9].

Reduction of membrane's dimension can be performed by considering a special, limiting, shape of the wave packet, such that one or more of membranes dimensions disappear. A wave packet which formally describes a state of an n -dimensional membrane, effectively describes a state of an $(n-1)$ -dimensional (or lower dimensional) membrane. This principle holds also for stationary states among which there are the conventional p -brane states.

This can be shown as follows. Suppose that $\psi[\tau, X^\mu(\xi^a)]$ is a wave functional of an n -dimensional membrane. Suppose now that it satisfies the relation

$$\frac{\delta\psi}{\delta X^\mu(\xi^{a_0}, \xi^i)} = \delta(\xi^{a_0} - \xi_\Sigma^{a_0})(\partial_{a_0} X^\mu \partial_{a_0} X_\mu)^{1/2} \frac{\delta\psi}{\delta X^\mu(\xi_\Sigma^{a_0}, \xi^i)} \quad (20)$$

where ξ^{a_0} is one of the coordinates ξ^a , $a = 1, 2, \dots, n$ and $\xi_\Sigma^{a_0}$ is its fixed value. Then $X^\mu(\xi_\Sigma^{a_0}, \xi^i) \equiv X^\mu(\xi^i)$, $i \neq a_0$ represents an $n-1$ dimensional membrane \mathcal{V}_{n-1} , and we have

$$\begin{aligned} D^{\mu(\xi)} D_{\mu(\xi)} \psi &= \int d^n \xi \frac{\Lambda}{\kappa \sqrt{|f|}} g^{\mu\nu} \frac{D^2 \psi}{DX^\mu(\xi) DX^\nu(\xi)} \\ &= \int d^{n-1} \xi \frac{\Lambda}{\kappa \sqrt{|\bar{f}|}} g^{\mu\nu} \frac{D^2 \psi}{DX^\mu(\xi^i) DX^\nu(\xi^i)} = \bar{D}^{\mu(\xi^i)} \bar{D}_{\mu(\xi^i)} \psi \end{aligned} \quad (21)$$

Here $\bar{f} \equiv \det \bar{f}_{ij} = f / \partial_{a_0} X^\mu \partial_{a_0} X_\mu$ is the determinant of the induced metric $\bar{f}_{ij} \equiv \partial_i X^\mu \partial_j X_\mu$ on \mathcal{V}_{n-1} .

Similarly

$$\Lambda^a \partial_a X^{\mu(\xi)} D_{\mu(\xi)} \psi = \Lambda^i \partial_i X^{\mu(\xi^i)} D_{\mu(\xi^i)} \psi \quad (22)$$

The expression (21) and (22) enter the Hamiltonian (11). The Schrödinger equation (14) which is formally an equation for a state $\psi[\tau, X^\mu(\xi^a)]$ of a membrane \mathcal{V}_n reduces to the equation of a state $\psi[\tau, X^\mu(\xi^i)]$ of a membrane \mathcal{V}_{n-1} . This process can be continued from $n-1$ to $n-2$, etc.

5 Conclusion

We embedded the theory of p -branes into a larger theory in which there is no constrained. Conventional p -brane states are particular stationary solutions to the covariant

Schrödinger equation in which an invariant evolution parameter τ takes place. We have generalized the Fock-Stueckelberg-Schwinger proper time formalism from point-particles to membranes. We adopt the interpretation that such a generalized theory (without constraints) has its physical content, and is not considered merely as a convenient mathematical tool.

Very useful is the concept of the membrane space \mathcal{M} which enables us to formulate the membrane's dynamics in essentially the same manner as the dynamics of the relativistic Stueckelberg point-particle. Construction of p -brane's field theory (to be presented elsewhere) seems to emerge uniquely and straightforwardly from this approach. Finally let us observe that the usual p -brane theories (with constraints) have not yet been fully confronted with experiments, therefore it makes sense to consider an enlarged theory. Fermions can also be considered by extending the formalism to include the Grassmann coordinates.

References

- [1] See e.g. E. Bergshoeff, E. Sezgin and P. K. Townsend, *Annals of Physics* 185 (1988) 330; M. Duff, *Class. Quant. Grav.* 6 (1989) 1577; M. Pavšič, *Phys. Lett.* B197 (1987) 327
- [2] G. W. Gibbons and D. L. Wiltshire, *Nucl. Phys.* B287 (1987) 717; T. Hori, *Phys. Lett.* B222 (1989) 188; M. Pavšič, *Class. Quant. Grav.* 2 (1985) 869; *Phys. Lett.* A116 (1986) 1; *Nuov. Cim.* A95 (1986) 297; *Found. Phys.* 24 (1994) 1495
- [3] M. Pavšič, *Found. Phys.* 26 (1996) 159
- [4] M. Pavšič, *Found. Phys.* 25 (1995) 819; *Nuov. Cim.* A108 (1995) 221
- [5] J. Hong, J. Kim and P. Sikivie: *Phys. Rev. Lett.* 69 (1992) 2611
- [6] V. Fock, *Phys. Z. Sowj.* 12 (1937) 404; E.C.G. Stueckelberg, *Helv. Phys. Acta* 14 (1941) 322; 14 (1941) 588; 15 (1942) 23; J. Schwinger, *Phys. Rev.* 82 (1951) 664; L. P. Horwitz and C. Piron, *Helv. Phys. Acta* 46 (1973) 316; E. R. Collins and J. R. Fanchi, *Nuovo Cimento* A48 (1978) 314
- [7] M. Pavšič, *Found. Phys.* 21 (1991) 1005; *Nuov. Cim.* A104 (1991) 1337
- [8] M. Pavšič, Preprint IJS-TP/96-10
- [9] A. Schild, *Phys. Rev.* D16 (1977) 1722; S. N. Roshchupkin and A. A. Zheltukhin, Preprint hep-th/9607119